Far East Journal of Theoretical Statistics

https://www.pphmjopenaccess.com
http://dx.doi.org/10.17654/0972086324001
Volume 68, Number 1, 2024, Pages 1-21 **P-ISSN: 0972-0863**

LAPLACE TRANSFORMATION FOR THE γ-ORDER GENERALIZED NORMAL, $N_\gamma(\mu, \sigma^2)$

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Abstract

We discuss a number of properties of the univariate γ-order generalized normal distribution, acting also as a solution to the heat equation. More emphasis is given on the Laplace transform of the introduced distribution. Logarithm Sobolev inequalities are discussed since they are the source of the introduced $N_\gamma(\mu, \Sigma)$.

Received: September 6, 2023; Revised: September 22, 2023; Accepted: October 10, 2023
2020 Mathematics Subject Classification: 62E15, 60E05, 62P35, 44A10.
Keywords and phrases: γ-order generalized normal distribution, Laplace transformation, heat equation, Sobolev inequality.

How to cite this article: Christos P. Kitsos and Ioannis S. Stamatiou, Laplace transformation for the γ-order generalized normal, $N_\gamma(\mu, \sigma^2)$, Far East Journal of Theoretical Statistics 68(1) (2024), 1-21. http://dx.doi.org/10.17654/0972086324001

Published Online: December 6, 2023

1. Introduction

In his early paper, Bliss [3] presented an integral inequality and the set of functions which turn inequality to equality. In this light some years later, while Sobolev was working on the problem of evaluation of the relation between the lower orders of the derivatives of a given function, with their upper orders, he came across the Sobolev Inequalities [20]. The impressive Sobolev inequality is:

$$\left(\int_{\mathbb{R}^p} | f(x)|^q dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^p} | \nabla f(x)|^2 dx\right)^{1/2} \qquad (1)$$

with $q = \dfrac{2p}{p-2}$ or in a compact form

$$\| f(x)\|_q \le C\| \nabla f \|_2 \qquad (2)$$

with p being the number of the involved variables (in Analysis the usual notation is n). Inequality (2) is valid for a differentiable function with compact support, ∇f is the gradient of f and the Sobolev constant C equals

$$C = \left(\frac{1}{\pi p(p-2)}\right)^{1/2}\left[\frac{\Gamma(p)}{\Gamma(p/2)}\right]^{2/p} > 0. \qquad (3)$$

Notice that in (1) equality holds if and only if

$$f(x) = \kappa(\lambda^2 + | x - r |^2)^{-q/2}$$

with $\kappa \in \mathbb{R}$, $\lambda > 0$, $r \in \mathbb{R}^p$. Notice that the Sobolev inequality can be defined also on a sphere, see [2].

The exponent q is crucial in (1), as only with such a q it holds, where the dimension $p > 2$. Moreover Sobolev proved that there exists a function embedding the Banach space $W^{m,p}(X)$ of the functions of $L_p(X)$ into $L_p(X)$ or to the space of continuous function $C(X)$, for particular m, p, q.

An important application of Logarithm Sobolev Inequality (LSI) is due to Markov Chains [8].

Nash's inequality is related to LSI. Moreover, by Nash's inequality [17], it is equivalent to evaluate the function $h(t, x, y)$ such that

$$\sup h(t, x, y) \leq Ct^{-p/2}, \quad \forall t > 0, \tag{4}$$

with $h(t, x, y)$ being the fundamental solution of the Cauchy problem. The function $h(t, x, y)$ is the solution of the "heat equation", see [11], for statistical line of thought, while see [22], from analysis point of view. This equals

$$h(t, x, y) = \frac{1}{(4\pi t)^{p/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{5}$$

The LSI was due to [21]. We recall the Gross logarithm inequality [9], with respect to the Gaussian weight

$$\int_{\mathbb{R}^p} |g|^2 \log|g|^2 dm \leq \frac{1}{\pi} \int_{\mathbb{R}^p} |\nabla g|^2 dm, \tag{6}$$

where

$$L^2(\mathbb{R}^p, dm) \ni \|g\|_2 = 1, \; dm = \exp\{-\pi x^2\} dx.$$

The Gross inequality (6) is equivalent, [23], to the Euclidean LSI as in (7):

$$\int_{\mathbb{R}^p} |g|^2 \log|g|^2 dm \leq \frac{p}{2} \log\left[\frac{2}{\pi pe} \int_{\mathbb{R}^p} |\nabla g|^2 dx\right] \tag{7}$$

with

$$g \in W^{1,2}(\mathbb{R}^p), \quad \int_{\mathbb{R}^p} |g|^2 dx = 1.$$

Relation (7) is very crucial, being in the order with the normal distribution, as the extremals of (7) with $g(x) = f(x, \mu, \sigma^2), \quad \sigma > 0,$

$\mu \in \mathbb{R}^p$, see [5, 14], are normal distributions. From (7) the γ-order generalized normal distribution emerged [14].

2. Probability Extensions - $N_\gamma(\mu, \Sigma)$

Following is the extension of [18] for LSI, as defined in (6) for $1 < \gamma < p$ and $f \in W^{1,\gamma}(\mathbb{R}^p)$ with $\| f \|_p = 1$ of the form

$$I(f, \gamma) \leq J(f, \gamma, \Lambda_\gamma)$$

with

$$I(f, \gamma) = \int_{\mathbb{R}^p} \| f \|^\gamma \log \| f \| dx,$$

$$J(f, \gamma, \Lambda_\gamma) = \frac{p}{\gamma^2} \log\left[\Lambda_\gamma \int_{\mathbb{R}^p} | \nabla f |^\gamma dx \right].$$

The optimal constant Λ_γ is

$$\Lambda_\gamma = \frac{\gamma}{p}\left(\frac{\gamma - 1}{e}\right)^{\gamma-1} \pi^{-\gamma/2} A^{\gamma/p}, \text{ where } A = A(p, \gamma) = \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(p\frac{\gamma-1}{\gamma} + 1\right)},$$

where Γ denotes the gamma function.

Consider $N_\gamma(\mu, \Sigma)$, see [14, 16], with position (mean) vector μ, positive definite scale parameter matrix $\Sigma \in \mathbb{R}^{p \times p}$, extra shape parameter $\gamma \in \mathbb{R} - [0, 1]$ and density function $\phi_\gamma(x; \mu, \Sigma)$ given by

$$\phi_\gamma(x; \mu, \Sigma) = C \exp\left\{ -\frac{\gamma - 1}{\gamma} [Q(x)]^{\frac{\gamma}{2(\gamma-1)}} \right\} \tag{8}$$

for $x \in \mathbb{R}^p$, where

$$Q(x) = (x - \mu)^T \Sigma^{-1} (x - \mu) \tag{9}$$

with

$$C = C(p, \gamma, \Sigma) = \frac{\left(\frac{\gamma - 1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}}}{\pi^{p/2}|\Sigma|^{1/2}} A(p, \gamma) \tag{10}$$

and $A = A(p, \gamma)$ as above. Figure 1 illustrates the pdf of $N_\gamma(0, I_p\sigma^2)$ when $p = 2$. See also [10] for a number of graphs of different values of γ. For an extensive analysis of $N_\gamma(\mu, I_p\sigma^2)$, see [15] for $p = 1$; and, for $p > 1$, see [16].

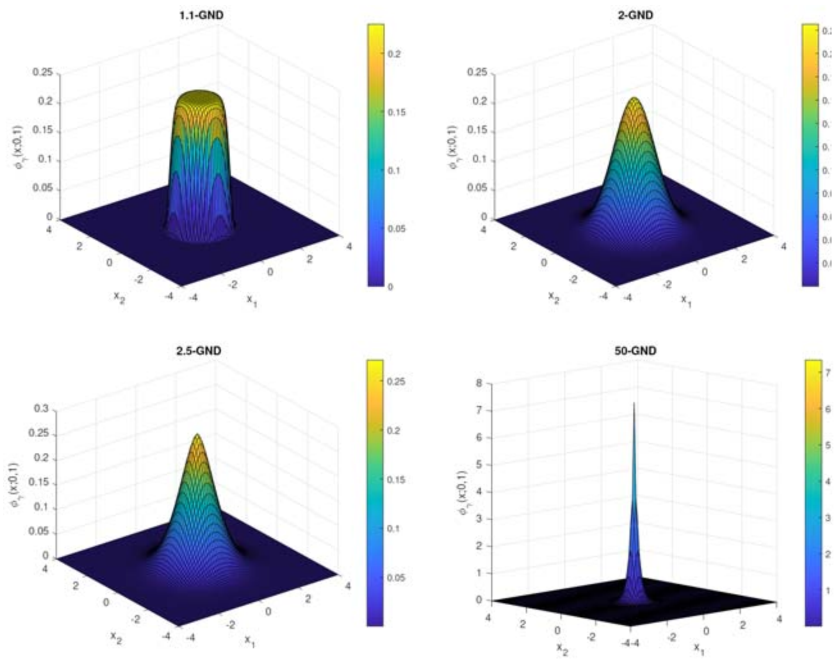

Figure 1. Plots of $\phi_\gamma(x; 0, I_p\sigma^2)$ for different values of γ, $p = 2$ and $\sigma = 1$.

The generalized γ-order multivariate normal distribution introduced by [14] has been discussed in detail by [16], while the univariate case was analyzed in [15]. In [16, Theorem 3.1], it has been proved that for $\gamma = 0$

$(\text{and } p = 1, 2)$ $\phi_\gamma(x)$ coincides with the Dirac distribution and for $\gamma = 1, 2, \pm\infty$ coincides with the uniform, normal and Laplace distributions, respectively. For $\gamma = 2,$ the well-known normal distribution is achieved, see also Figure 2.

Furthermore, the classical entropy inequality [6], can be extended to

$$\left[\frac{2\pi e}{p} Var(X)\right]^{1/2} \left[\Lambda_\gamma \frac{1}{\gamma^\gamma} J_\gamma(X)\right]^{1/\gamma} \geq 1. \tag{11}$$

Actually $J_\gamma(X)$ represents an extension of Fisher's entropy type information measure

$$\begin{aligned} J_\gamma(X) &= \int_{\mathbb{R}^p} |\nabla \log f|^\gamma dx \\ &= \int_{\mathbb{R}^p} |\nabla f|^\gamma f^{1-\gamma} dx \end{aligned} \tag{12}$$

as with $\gamma = 2,$

$$\begin{aligned} J_2(X) &= \int_{\mathbb{R}^p} |\nabla \log f|^2 dx \\ &= \int_{\mathbb{R}^p} (\nabla f)(\nabla \log f) dx = J(X) \end{aligned} \tag{13}$$

see [14].

The heat equation [11, Chapter 7] can be generalized through the $\phi_\gamma(x)$ distribution as follows: Consider a standard Brownian motion $\{X(t); t > 0\}$ coming from $N_\gamma(0, t),$ i.e., from the γ-order generalized normal distribution with density function

$$\phi_\gamma(x; 0, t) = \frac{\lambda_\gamma}{\sqrt{\pi t}} \exp\left\{-\frac{\gamma - 1}{\gamma}\left(\frac{x}{\sqrt{t}}\right)^{\frac{\gamma}{\gamma-1}}\right\} \tag{14}$$

with

$$\lambda_\gamma = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{\gamma-1}{\gamma}+1\right)}\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}, \tag{15}$$

see [13], for details.

Theorem 1 (Kitsos [13])**.** *There exists a well defined function* $K = K(x;\, t,\, \gamma)$ *such that*

$$\frac{\partial^2 \phi_\gamma}{\partial x^2} = K\,\frac{\partial \phi_\gamma}{\partial t} \tag{16}$$

with $K = K(x;\, t,\, \gamma) = \dfrac{N(x;\, t,\, \gamma)}{D(x;\, t,\, \gamma)}$, *where*

$$N(\cdot) = t^{-\gamma_1} x^{\frac{2}{\gamma-1}} - \frac{1}{\gamma-1} t^{-\frac{1}{2}\gamma_1} x^{\frac{2-\gamma}{\gamma-1}},$$

$$D(\cdot) = \frac{1}{2}\left(-\frac{1}{t} + \frac{x^{\gamma_1}}{t^{\frac{3\gamma-2}{2(\gamma-1)}}}\right), \quad \gamma_1 = \frac{\gamma}{\gamma-1}. \tag{17}$$

For $t = 1$, we find that $D(x;\, 1,\, \gamma) = \frac{1}{2}(-1 + x^{\gamma_1})$ and therefore $K(x;\, 1,\, \gamma)$ is defined for $x^{\gamma_1} \neq 1$ and in principle $x \neq 1$. See Figure 3 below, for a graphical representation of $K(x;\, 1,\, \gamma)$ for various values of γ, where a special consideration is needed for the *x*-values at MATLAB. Notice that, see [13, Corollary 4.1], for $\gamma = 2$,

$$K = K(x;\, t,\, 2) = 2 \tag{18}$$

and therefore (16) is reduced to the classical heat equation [11]:

$$\frac{\partial^2 \phi_2}{\partial x^2} = 2\,\frac{\partial \phi_2}{\partial t}. \tag{19}$$

As far as the corresponding values of $K(x; 1, \gamma)$ are concerned for $\gamma = 2,$ the γ-order generalized normal distribution is reduced to normal and $K = 2,$ while for $\gamma \downarrow 1,$ $K(0; 1, 1) = 0$ and for $x \in (0 - \varepsilon, 0 + \varepsilon),$ $K(x; 1, 1)$ is constant, see Figure 3.

Recall that for $\gamma \downarrow 1$ and $p = 1, \mu = 0, t = 1 = \sigma_t,$ the defined $\phi_\gamma(x)$ distribution as in (14) approaches the uniform distribution while for $\gamma \to \infty$ the $\phi_\gamma(x)$ distribution approaches the Laplace distribution, [12].

3. Laplace Transform of the $N_\gamma(\mu, \sigma^2)$

For $p = 1,$ the Laplace transform of $\phi_\gamma(x; \mu, \sigma^2)$ is defined as

$$\mathcal{L}\phi_\gamma(\xi) = \int_{-\infty}^{\infty} \exp\{\xi x\}\phi_\gamma(x; \mu, \sigma^2)dx. \tag{20}$$

Equation (20) characterizes the r.v. X uniquely in the sense that the p.d.f. of X can be recovered by taking the inverse Laplace transform in (20). The following result provides expression for (20).

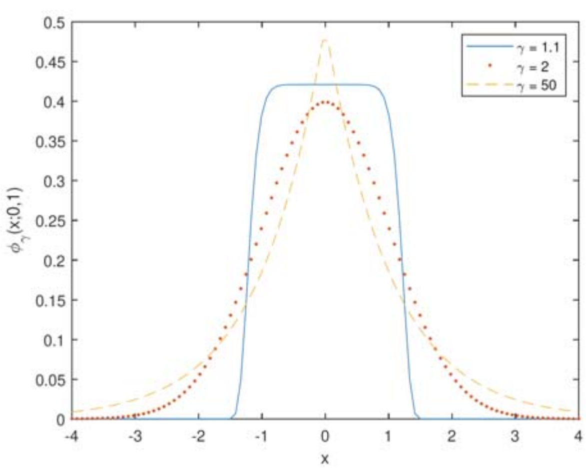

Figure 2. Plots of the univariate $\phi_\gamma(x; 0, 1)$ for different values of γ.

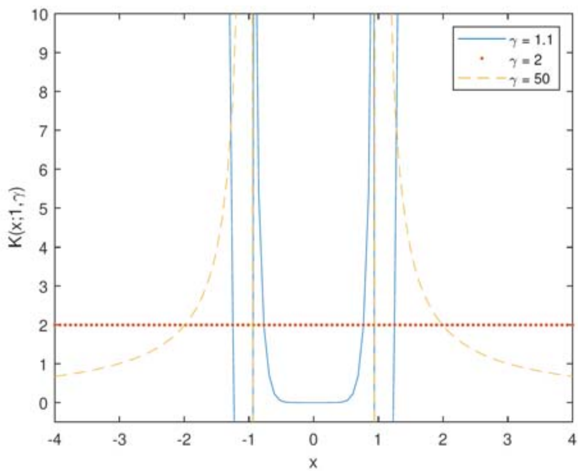

Figure 3. Plots of $K(x;\, 1,\, \gamma)$ for different values of γ.

Theorem 2. *The Laplace transform of* $\phi_\gamma(x;\, \mu,\, \sigma^2)$ *with* $p = 1$ *reads*

$$\mathcal{L}\phi_\gamma(\xi) = \gamma_0 \frac{e^{\xi\mu}}{\Gamma(\gamma_0 + 1)} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma(\gamma_1)^{\gamma_0})^{2j} \Gamma((2j+1)\gamma_0), \tag{21}$$

where $\gamma_0 = \dfrac{\gamma - 1}{\gamma}$ *and* $\gamma_1 = \dfrac{1}{\gamma_0}.$

Proof. We rewrite (20) using (8) as

$$\mathcal{L}\phi_\gamma(\xi) = C\int_{-\infty}^{\infty} \exp\left\{\xi x - \frac{\gamma-1}{\gamma}[Q(x)]^{\frac{\gamma}{2(\gamma-1)}}\right\} dx$$

$$= C\int_{-\infty}^{\infty} \exp\left\{\xi x - \frac{\gamma-1}{\gamma}\left|\frac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\} dx$$

$$= C\sigma e^{\xi\mu}\int_{-\infty}^{\infty} \exp\left\{\xi\sigma z - \frac{\gamma-1}{\gamma}|\, z\,|^{\frac{\gamma}{\gamma-1}}\right\} dz,$$

where the typical variable transform is given by $z = (x - \mu)/\sigma.$ Therefore the evaluation of the integral

$$I(\xi) := \int_0^{\infty} \exp\left\{\xi\sigma z - \frac{\gamma-1}{\gamma} z^{\frac{\gamma}{\gamma-1}}\right\} dz$$

is needed, since then

$$\mathcal{L}\phi_\gamma(\xi) = C\sigma e^{\xi\mu}\left(\int_{-\infty}^{0} \exp\left\{\xi\sigma z - \frac{\gamma-1}{\gamma}|z|^{\frac{\gamma}{\gamma-1}}\right\} dz\right.$$

$$\left. + \int_0^{\infty} \exp\left\{\xi\sigma z - \frac{\gamma-1}{\gamma}|z|^{\frac{\gamma}{\gamma-1}}\right\} dz\right)$$

$$= C\sigma e^{\xi\mu}\int_0^{\infty} \exp\left\{-\xi\sigma z - \frac{\gamma-1}{\gamma} z^{\frac{\gamma}{\gamma-1}}\right\} dz$$

$$+ C\sigma e^{\xi\mu}\int_0^{\infty} \exp\left\{\xi\sigma z - \frac{\gamma-1}{\gamma} z^{\frac{\gamma}{\gamma-1}}\right\} dz$$

so due to the definition of $I(\xi)$, the above relation is reduced to

$$\mathcal{L}\phi_\gamma(\xi) = C\sigma e^{\xi\mu}[I(\xi) + I(-\xi)].$$

Now, due to the definition of the constant term, see also (10), we have

$$\mathcal{L}\phi_\gamma(\xi) = \frac{\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}}{\pi^{1/2}} \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{\gamma-1}{\gamma}+1\right)} e^{\xi\mu}(I(\xi) + I(-\xi)). \tag{22}$$

Writing the exponential in the integral as

$$e^{\xi\sigma z} = \sum_{j=0}^{\infty} \frac{(\xi\sigma z)^j}{j!}$$

and using Fubini's theorem, we have

$$I(\xi) = \int_0^\infty \sum_{j=0}^\infty \frac{(\xi\sigma z)^j}{j!} \exp\left\{-\frac{\gamma-1}{\gamma} z^{\frac{\gamma}{\gamma-1}}\right\} dz$$

$$= \sum_{j=0}^\infty \frac{(\xi\sigma)^j}{j!} \int_0^\infty z^{j+1-1} \exp\left\{-\frac{\gamma-1}{\gamma} z^{\frac{\gamma}{\gamma-1}}\right\} dz$$

$$= \sum_{j=0}^\infty \frac{(\xi\sigma)^j}{j!} \frac{1}{\gamma_1} \left(\frac{1}{\gamma_0}\right)^{\frac{j+1}{\gamma_1}} \Gamma\left(\frac{(j+1)}{\gamma_1}\right)$$

$$= \left(\frac{1}{\gamma_0}\right)^{\gamma_0 - 1} \sum_{j=0}^\infty \frac{1}{j!} \left(\xi\sigma\left(\frac{1}{\gamma_0}\right)^{\gamma_0}\right)^j \Gamma((j+1)\gamma_0),$$

where we have also used that

$$\int_0^\infty x^{s-1} e^{-\gamma_0 x^{\gamma_1}} dx = \frac{1}{\gamma_1}\left(\frac{1}{\gamma_0}\right)^{\frac{s}{\gamma_1}} \Gamma\left(\frac{s}{\gamma_1}\right) \tag{23}$$

for $s = j + 1$, $\gamma_0 = (\gamma - 1)/\gamma$, $\gamma_1 = 1/\gamma_0$. Therefore by adding $I(\xi)$ and $I(-\xi)$ and using (22), we get the representation

$$\mathcal{L}\phi_\gamma(\xi) = 2\gamma_0 \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma(\gamma_0+1)} \frac{e^{\xi\mu}}{\sqrt{\pi}} \sum_{j=0}^\infty \frac{1}{(2j)!} (\xi\sigma(\gamma_1)^{\gamma_0})^{2j} \Gamma((2j+1)\gamma_0),$$

or by using the value $\Gamma(3/2) = \sqrt{\pi}/2$, we have

$$\mathcal{L}\phi_\gamma(\xi) = \gamma_0 \frac{e^{\xi\mu}}{\Gamma(\gamma_0+1)} \sum_{j=0}^\infty \frac{1}{(2j)!} (\xi\sigma(\gamma_1)^{\gamma_0})^{2j} \Gamma((2j+1)\gamma_0),$$

which is exactly (21). □

Corollary 1. *When* $\gamma = 2$, *the Laplace transform of the classical normal distribution* $N(\mu, \sigma^2)$ *is obtained as*

$$\mathcal{L}\phi_2(\xi) = \exp\left\{\xi\mu + \frac{\xi^2\sigma^2}{2}\right\}.$$

Proof. By (21) for $\gamma = 2$,

$$\mathcal{L}\phi_2(\xi) = \frac{e^{\xi\mu}}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma 2^{1/2})^{2j} \Gamma\left(\frac{(2j+1)}{2}\right)$$

$$= \frac{e^{\xi\mu}}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi^2\sigma^2 2)^{j} \Gamma\left(j + \frac{1}{2}\right)$$

$$= \frac{e^{\xi\mu}}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi^2\sigma^2 2)^{j} \frac{(2j)!}{4^j j!} \sqrt{\pi}$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\xi^2\sigma^2}{2}\right)^{j}$$

$$= \exp\left\{\xi\mu + \frac{\xi^2\sigma^2}{2}\right\}$$

which is indeed the moment generating function of the normal distribution. □

Corollary 2. *When* $\gamma \to \pm\infty$, *the Laplace transform of the Laplace distribution with parameters* μ, σ *is obtained as*

$$\mathcal{L}\phi_\infty(\xi) = e^{\xi\mu} \frac{1}{1 - \xi^2\sigma^2}.$$

Proof. Letting $\gamma \to \pm\infty$ in (21), $\mathcal{L}\phi_\gamma(\xi)$ tends to $\mathcal{L}\phi_\infty(\xi)$, where

$$\mathcal{L}\phi_\infty(\xi) = e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma)^{2j} \Gamma(2j+1)$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi^2\sigma^2)^{j} (2j)!$$

$$= e^{\xi\mu} \frac{1}{1 - \xi^2\sigma^2}$$

for all ξ such that $|\xi| < \dfrac{1}{\sigma}$, which is the moment generating function of the Laplace distribution. □

In the following Theorems 3 and 4, we introduce one more representation of the Laplace transform of $N_\gamma(\mu, \sigma^2)$ adopting the Beta function and an upper bound in compact form in terms of expectation.

Theorem 3. *The Laplace transform of* $N_\gamma(\mu, \sigma^2)$ *is equivalent to*

$$\mathcal{L}\phi_\gamma(\xi) = e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0}\Gamma(\gamma_0))^{2j} \prod_{m=1}^{2j} \frac{1}{B(\gamma_0, m\gamma_0)}, \tag{24}$$

when γ_0 *is integer* (*in cases where* $\gamma = -1/k,\ k \in \mathbb{N}$).

Proof. We turn once more to (21) in order to get a more suitable expression. By (21), we have

$$\mathcal{L}\phi_\gamma(\xi) = e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0})^{2j} \frac{\Gamma((2j+1)\gamma_0)}{\Gamma(\gamma_0)}$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0})^{2j} \frac{\Gamma((2j+1)\gamma_0)}{(\Gamma(\gamma_0))^{2j+1}} (\Gamma(\gamma_0))^{2j}$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0}\Gamma(\gamma_0))^{2j} [B(\gamma_0, \gamma_0, \dots, \gamma_0)]^{-1},$$

where B is the multivariate Beta function with $2j+1$ arguments. We may write

$$B(\gamma_0, \gamma_0, \dots, \gamma_0)$$

$$= \int_0^1 y_1^{\gamma_0 - 1}(1 - y_1)^{2j\gamma_0 - 1} dy_1 \int_0^1 y_2^{\gamma_0 - 1}(1 - y_2)^{(2j-1)\gamma_0 - 1} dy_2$$

$$\cdots \int_0^1 y_{2j-1}^{\gamma_0 - 1}(1 - y_{2j-1})^{2\gamma_0 - 1} dy_{2j-1} \int_0^1 y_{2j}^{\gamma_0 - 1}(1 - y_{2j})^{\gamma_0 - 1} dy_{2j}$$

$$= \prod_{m=1}^{2j} \int_0^1 y^{\gamma_0 - 1}(1 - y)^{m\gamma_0 - 1} dy$$

$$= \prod_{m=1}^{2j} B(\gamma_0, m\gamma_0).$$

Therefore, we express the Laplace transform (21) in the form (24). □

Theorem 4. *The Laplace transform of* $N_\gamma(\mu, \sigma^2)$ *in terms of the exponential moments of the standardized* $N_\gamma(0, 1)$ *is bounded in the following way*

$$\mathcal{L}\phi_\gamma(\xi) \leq e^{\xi\mu} \mathbb{E}\left(\exp\left\{ \frac{\xi^2 \sigma^2 Z^2}{2} \right\} \right), \tag{25}$$

where $Z \sim N_\gamma(0, 1)$ *and* $\sigma^2 Z \sim N_\gamma(0, \sigma^2)$.

Proof. Let $Z \sim N_\gamma(0, 1)$, that is the r.v. Z has the law of the standard univariate $N_\gamma(0, 1)$, with pdf $\phi_\gamma(x; 0, 1)$. Then

$$\mathcal{L}\phi_\gamma(\xi) = e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0})^{2j} \frac{\Gamma((2j+1)\gamma_0)}{\Gamma(\gamma_0)}$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma\gamma_0^{-\gamma_0})^{2j} \mathbb{E}(\gamma_0^{2j\gamma_0} Z^{2j})$$

$$= e^{\xi\mu} \sum_{j=0}^{\infty} \frac{1}{(2j)!} (\xi\sigma)^{2j} \mathbb{E}(Z^{2j})$$

$$\leq e^{\xi\mu} \mathbb{E}\left(\sum_{j=0}^{\infty} \frac{1}{2^j j!} (\xi\sigma Z)^{2j} \right)$$

$$= e^{\xi\mu}\mathbb{E}\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\xi^2\sigma^2 Z^2}{2}\right)^j\right),$$

where, in the second step, we have used [15, Lemma 2.1], and in the fourth step, the inequality $(2j)! \geq 2^j j!$. The last equation implies (25). □

Inequality (25) turns to equality for the limiting cases of $\gamma \uparrow 0$ and $\gamma \downarrow 1$. Indeed, we have

Corollary 3. *When* $\gamma \uparrow 0$, *the Laplace transform of the Dirac distribution* $D(\mu)$ *is obtained as follows*:

$$\mathcal{L}\phi_0(\xi) = \exp\{\xi\mu\}.$$

Proof. Recall (25) and the fact that $N_\gamma(0, 1)$ becomes the Dirac distribution $D(0)$ as $\gamma \uparrow 0$, see [16]. Then taking the limit as $\gamma \uparrow 0$ in the R.H.S. of (25), we have

$$e^{\xi\mu}\int_{-\infty}^{\infty}\exp\left\{\frac{\xi^2\sigma^2u^2}{2}\right\}\delta(u)du = e^{\xi\mu}\int_{-\varepsilon}^{+\varepsilon}\exp\left\{\frac{\xi^2\sigma^2u^2}{2}\right\}\delta(u)du$$

$$= \exp\{\xi\mu\}$$

for $\varepsilon > 0$ which is the Dirac distribution $D(\mu)$, while also

$$\lim_{\gamma\uparrow 0}\mathcal{L}\phi_\gamma(\xi) = \mathcal{L}\phi_0(\xi) = \mathcal{L}D(\mu)(\xi) = \exp\{\xi\mu\}.$$ □

Corollary 4. *When* $\gamma \downarrow 1$, *the Laplace transform of the uniform distribution with parameters* $(\mu - \sigma, \mu + \sigma)$ *is obtained as follows*:

$$\mathcal{L}\phi_1(\xi) = \frac{1}{2}e^{\xi\mu}\frac{e^{\xi\sigma} - e^{-\xi\sigma}}{\xi\sigma}.$$

Proof. Recall (25) and the fact that $N_\gamma(0, 1)$ becomes the uniform distribution in $(-1, 1)$ as $\gamma \downarrow 1$, see [16]. Then taking the limit as $\gamma \downarrow 1$ in the R.H.S. of (25), we have

$$e^{\xi\mu}\int_{-\infty}^{\infty} \exp\left\{\frac{\xi^2\sigma^2u^2}{2}\right\}\frac{1}{2}\,du = \frac{1}{2}e^{\xi\mu}\int_{-1}^{1} \exp\left\{\frac{\xi^2\sigma^2u^2}{2}\right\}du$$

$$= \frac{1}{2\xi\sigma}e^{\xi\mu}\sqrt{2}\int_{-\xi\sigma/\sqrt{2}}^{\xi\sigma/\sqrt{2}} e^{x^2}\,dx. \tag{26}$$

Note that

$$\int e^{x^2}dx = -i\frac{\sqrt{\pi}}{2}erf(ix) = \frac{\sqrt{\pi}}{2}erfi(x),$$

where $erf(\cdot)$ and $erfi(\cdot)$ are the error function and complex error function accordingly and i the imaginary unit so

$$\int_{-\xi\sigma/\sqrt{2}}^{\xi\sigma/\sqrt{2}} e^{x^2}dx = -i\frac{\sqrt{\pi}}{2}(erf(i\xi\sigma/\sqrt{2}) - erf(-i\xi\sigma/\sqrt{2}))$$

$$= -i\frac{\sqrt{\pi}}{2}2i\mathcal{I}(erf(i\xi\sigma/\sqrt{2})) = \sqrt{\pi}\mathcal{I}(erf(i\xi\sigma/\sqrt{2})), \tag{27}$$

where $\mathcal{I}(\cdot)$ denotes the imaginary part of a complex number. Note that we have used $erf(\bar{z}) = \overline{erf(z)}$. Moreover,

$$erf(i\xi\sigma/\sqrt{2}) = \frac{2}{\sqrt{\pi}}\int_{0}^{i\xi\sigma/\sqrt{2}} e^{-x^2}dx = 2\left(\Phi(i\xi\sigma) - \frac{1}{2}\right), \tag{28}$$

where $\Phi(\cdot)$ is the c.d.f. of the standard normal. Finally, using

$$\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)2^k k!}$$

at $t = i\xi\sigma$, (27) and (28) in (26), we obtain

$$\frac{1}{2\xi\sigma} e^{\xi\mu} \sqrt{2}\sqrt{\pi}\mathcal{I}\left(2\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{(i\xi\sigma)^{2k+1}}{(2k+1)2^k k!} \right)$$

$$= \frac{1}{2\xi\sigma} e^{\xi\mu} \mathcal{I}\left(2\sum_{k=0}^{\infty} \frac{(\xi\sigma)^{2k+1}}{(2k+1)!} (-1)^k (i)^{2k+1} \right)$$

$$= \frac{1}{2\xi\sigma} e^{\xi\mu} 2\sum_{k=0}^{\infty} \frac{(\xi\sigma)^{2k+1}}{(2k+1)!} \mathcal{I}((-1)^k (i)^{2k+1})$$

$$= \frac{1}{2\xi\sigma} e^{\xi\mu} \sum_{k=0}^{\infty} \frac{(\xi\sigma)^k - (-\xi\sigma)^k}{k!} \mathcal{I}((-1)^k (i)^{2k+1})$$

$$= \frac{1}{2\xi\sigma} e^{\xi\mu} (e^{\xi\sigma} - e^{-\xi\sigma}), \tag{29}$$

where it has been used that $\mathcal{I}((-1)^k (i)^{2k+1}) = 1$ which is easy to show by distinguishing odd and even values of *k*. To conclude notice that the Laplace transform $\mathcal{L}\phi_1(\xi)$, of the uniform distribution in $(\mu - \sigma, \mu + \sigma)$, can be easily obtained as:

$$\lim_{\gamma \downarrow 1} \mathcal{L}\phi_\gamma(\xi) = \mathcal{L}\phi_1(\xi) = \int_{\mu-\sigma}^{\mu+\sigma} e^{\xi x} \frac{1}{2\sigma} dx$$

$$= \frac{1}{2\xi\sigma} (e^{\xi(\mu+\sigma)} - e^{\xi(\mu-\sigma)}) = \frac{e^{\xi\mu}}{2\xi\sigma} (e^{\xi\sigma} - e^{-\xi\sigma}),$$

which is the same as (29). □

Corollary 5. *The Laplace transforms of the uniform, normal and Laplace distributions are functions of the Laplace transform of the Dirac.*

The above results are collected in Table 1, with other values γ around 2 and 3 to compare "fat-tailed" distributions. The calculations have been proceeded with MATLAB. These provide evidence that the Laplace transform can be easily calculated for the univariate γ-order generalized normal distribution, due to Theorem 2 for any given value of γ.

4. Discussion

The Weak Law of large numbers, by Bernoulli in 1713, was the first step towards the Probability Theory. Later on, in 1733, the normal distribution appeared as an approximation to the probability for sums of Binomial distributed quantities, to be in between two given values, by de Moivre, as the spiritual research in [7]. It was Gauss in 1809 in his "Theoria Motus Corponum Coelestim" stating the Least Squares, known to him since 1795, declaring that the model was appropriate when the "errors" were coming from a normal distribution (in current terminology) [19]. The hypothesis of errors were established and adapted especially by the astronomers [1].

Table 1. Laplace distribution of $N_\gamma(\mu, \sigma^2)$ for different values of γ and μ, σ

γ	Distribution	$\mathcal{L}\phi_\gamma(\xi)$
0	Dirac (μ)	$\exp\{\xi\mu\}$
0	Dirac (0)	1
1	Uniform $(\mu - \sigma, \mu + \sigma)$	$\frac{1}{2\xi\sigma}\exp\{\xi\mu\}(e^{\xi\sigma} - e^{-\xi\sigma})$
1	Uniform $(-1, 1)$	$\frac{1}{2\xi}(e^{\xi} - e^{-\xi})$
1.9	$N_{1,9}(0, 1)$	$0.5348 + 0.4811\xi^2 + 0.1117\xi^4 + 0.0169\xi^6$
2	Normal (μ, σ^2)	$\exp\left\{\xi\mu + \frac{\xi^2\sigma^2}{2}\right\}$
2	Normal $(0, 1)$	$\exp\left\{\frac{\xi^2}{2}\right\}$
2.2	$N_{2,2}(0, 1)$	$0.6139 + 0.5340\xi^2 + 0.1516\xi^4 + 0.0297\xi^6$
2.4	$N_{2,4}(0, 1)$	$0.6541 + 0.5638\xi^2 + 0.1778\xi^4 + 0.0400\xi^6$
3	$N_3(0, 1)$	$0.7385 + 0.6340\xi^2 + 0.2520\xi^4 + 0.0764\xi^6$
3.5	$N_{3,5}(0, 1)$	$0.7837 + 0.6775\xi^2 + 0.3073\xi^4 + 0.1105\xi^6$
$\pm\infty$	Laplace (μ, σ)	$\exp\{\xi\mu\}/(1 - \xi^2\sigma^2)$
$\pm\infty$	Laplace $(0, 1)$	$1/(1 - \xi^2)$

The Gaussian distribution has been generalized, as in Section 2, due to the introduced parameter γ, refer [15, 16] also. So in this paper γ-order generalized normal distribution is discussed as an extension of the normal distribution with mean μ and variance σ^2, introducing an extra shape parameter. This distribution has emerged as an external from LSI, [14] and helps for the use of LSI in Statistics.

The LSI appears, recently, to be applied in a number of applications related with uncertainty and Statistical Information Theory [12]. We introduced LSI in a compact form and investigated some essentials, to our consideration in areas of application. More extensions can be obtained such as the Blachman-Stam inequality [14], while a number of nice theoretical results can be obtained [4], among others. Not only the theoretical insight is covered with the distribution introduced in Section 2, but it offers a model to approach the fat tailed distributions [16, Table 1 and 2]. As a continuation of our work, Table 1 above provides evidence that the Laplace transformation can be evaluated for all the real values of the shape parameter γ, but not within $[0, 1]$.

The Laplace transformation for the $N_\gamma(\mu, \sigma^2)$ distribution was introduced and related results were obtained. We are referring to Laplace transformation rather than the moment generating function as the Laplace transformation. We are planning the same to apply in our future work. Despite the fact that (21) might be considered complicated, still with MATLAB nice results can be obtained for the “fat-tailed distribution”, as can be considered and evaluated easily, with Table 1 providing the appropriate evidence.

Acknowledgments

The first author would like to thank the University of Aberta, Lisbon, Portugal for offering the chance to lecture for the academic year 2023-24 for the PhD program DNAM.

The authors are highly grateful to the referee for his careful reading, valuable suggestions and comments, which helped to improve the paper.

Also, the authors would like to thank the “Special Account for Research Grants (ELKE)” of the University of West Attica who kindly eventually supported us.

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